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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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invariant points of the transformation, namely, those points that are their own homologues. Hence

$$x : y : z = \phi_1 : \phi_2 : \phi_3.$$

So an invariant point is the intersection of

$$x\phi_3 - z\phi_1 = 0$$

and

$$y\phi_3 - z\phi_2 = 0,$$

other than the F-system and the n intersections of z and ϕ_3. Hence in general there are

$$(n + 1)^2 - (n^2 - 1) - n = n + 2$$

invariant points. If the two $(n + 1)$-ics have a common part it will be a curve of fixed points for the transformation and the number of isolated invariant points will be reduced.

If μ be the degree of the invariant curve, the number of isolated nvariant points is *

$$n + 2 + \mu^2 - 3\mu - \Sigma\,\delta_\alpha^{\,2}$$

where δ_α is the order of multiplicity of the F-point O_α on the invariant curve. In the case when $\mu = 2$, the number of isolated invariant points becomes $n - \Sigma\,\delta_\alpha^2$. Also the invariant curve f_2 must pass through the F-points with total multiplicity $(2n - 2)$. Further, when $n = 4$, there are two cases of the non-De Jonquiere's type,† namely, (i) when the invariant conic passes through the three double F-points, (ii) when the conic passes through two double F-points and two single F-points. In the first case there is one isolated invariant point and in the second none.

De Jonquiere's transformation T_j is a special type in which there is an F-point of maximum multiplicity $n - 1$ and $(2n - 2)$ simple F-points. The multiple F-point is called the vertex O of the transformation. In this, the pencil of lines through O transforms into another pencil of lines; and conversely if two pencils of lines correspond, the transformation is a T_j for the vertices are $(n - 1)$-ple F-points.

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* Hudson : *loc. cit.*, p. 87.

† The Jonquiere's type is discussed in the most general case and hence not considered in the particular case.

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2. Transformation with a Conic of fixed points.

The general transformation for which every point of a conic S is a fixed point is of the form

$$\left.\begin{aligned} \rho x' &= x f_n(x, y, z) + S(x, y, z) . P . (x, y, z) \\ \rho y' &= y f_n(x, y, z) + S(x, y, z)\ Q(x, y, z) \\ \rho z' &= z f_n(x, y, z) + S(x, y, z)\ R(x, y, z) \end{aligned}\right\} \quad \ldots \quad (3)$$

where f_n is of order n and P, Q, R, each of order $n-1$. For this to be a Cremona transformation f_n, P, Q, R must satisfy further conditions, namely, that the three expressions on the right of (3) equated to zero must define a homoloidal net of curves. These further conditions can be easily determined for small values of n. The points common to S and f_n are the $2n$ base points of the net and the remaining F-points are the points common to $xf_n + SP = 0$ and $yf_n + SQ = 0$ lying on $zf_n + SR = 0$. Hence the remaining base points not lying on S lie on the curves $yP - xQ = 0$ and $zP - xR = 0$, namely, on

$$P/x = Q/y = R/z.$$

Before discussing the case for $n = 4$, I shall consider the simple case of $n = 2$ and the general Jonquiere's type.

3. Quadratic transformation T_2, with a Conic of fixed points.

Since there are only three simple F-points for each system for a quadratic transformation, in order that S may be a conic of fixed points for this transformation, two pairs of F-points, one from each system, must lie on S. Suppose O_1, O_2; O'_1, O'_2 are the points on S, and the other F-points O_3, O'_3 are outside it. A general line l, through O_3 will be transformed into a general line l' through O'_3 and since l meets S in two fixed points D_1D_2, l' will pass through D_1D_2 and hence coincide with l. Hence $O_3 \equiv O'_3$ and is the centre of the transformation. The projective pencils coincide $O_3O_1 \equiv O'_3O'_2$ and similarly $O_3O_2O'_1$ are collinear.* This T_2 will be a central involution if $O_1 \equiv O'_2$ and $O_2 \equiv O'_1$. Corresponding points are conjugate points with regard to

* Hudson : *loc. cit.*, p. 47.

the conic and are collinear with the fixed point O_3. The equations of the transformation can be easily obtained from the set of equations (3).

If $x^2 + y^2 + z^2 = 0$ is the equation of the conic S referred to a self-polar triangle, the general equations of transformation take the form :

$$\left.\begin{aligned} \rho x' &= x(lx + my + nz) + \lambda(x^2 + y^2 + z^2) \\ \rho y' &= y(lx + my + nz) + \mu(x^2 + y^2 + z^2) \\ \rho z' &= z(lx + my + nz) + \nu(x^2 + y^2 + z^2) \end{aligned}\right\} \quad \ldots \quad (4)$$

If the three right-hand members equated to zero generate a homoloidal net, the three base points of the net which are the F-points of the transformation are the two points of intersection of the line $lx + my + nz = 0$ and the conic $x^2 + y^2 + z^2 = 0$ and the point $(\lambda \mu \nu)$ given by

$$\frac{\lambda}{x} = \frac{\mu}{y} = \frac{\nu}{z}.$$

Since $(\lambda \mu \nu)$ should lie on each of the curves of the net we have

$$\lambda(l + \lambda) + \mu(m + \mu) + \nu(n + \nu) = 0 \quad \ldots \quad (5)$$

Thus, if $\lambda \mu \nu$ satisfy this relation, the equations given by (4) are those of a quadratic transformation with a conic of fixed points. It is easily seen that a number of such transformations can be obtained subject to the condition (5).

A very familiar form of the transformation is got by taking $(-l, -m, -n)$ for (λ, μ, ν). Then the equations of transformation become

$$\begin{aligned} \rho x' &= x(my + nz) - l(y^2 + z^2) \\ \rho y' &= y(lx + nz) - m(x^2 + z^2) \\ \rho z' &= z(lx + my) - n(x^2 + y^2) \end{aligned}$$

The point (l, m, n) is the pole of the line $lx + my + nz = 0$ with respect to the fixed conic S and pairs of corresponding points are collinear with O (l, m, n). This transformation is ordinary inversion when the conic is a circle with centre O. Hence the transformation is termed generalised inversion. Also it is an involutoric transformation.

A different type of transformation is obtained by taking a set of values for λ, μ, ν given by

$$\lambda^2 + \mu m = 0, \quad \mu^2 + \lambda l = 0, \qquad \nu + n = 0$$

i.e., the values $(\sqrt[3]{-m^2 l},\ \sqrt[3]{-l^2 m}, -n)$. The equations of transformation are easily got and the transformation is a non-involutoric one.

A projective treatment of the transformation is given by L. F. Dickson.* Suppose given a conic S and two pairs O_1, O'_1; O_2, O'_2 of its points as associate F-points. We construct the transform P' of a point P in the transformation with the given F-points and S for a conic of fixed points, by joining O_1P, O_2P to meet S in X_1, X_2 and again joining O'_1X_1, O'_2X_2 to intersect at P'. It is easily seen that $O_3 PP'$ is the Pascal line of the hexagon $O_1X_1O'_1\ O_2X_2O'_2$. The third F-point of the transformation is $O_3 \equiv O'_3$. We will refer to this construction as Dickson's construction for a central T_2.

4. Composition of Transformations of order *n*.†

Now I will consider how a general Cremona transformation of order *n* can be obtained as the product of a number of quadratic transformations. I am considering only those types that have an application in the present investigation; the other types will be found discussed in Hudson.‡

Consider two quadratic transformations V and W. Let the F-points of V be O_{11}, O_{12}, O_{13} and O'_{11}, O'_{12} O'_{13} in the two systems; and those of W be $O'_{21}\,O'_{22}\,O'_{23}$ and $O''_{21}\,O''_{22}\,O''_{23}$. Also let O'_{21} be the transform of O_{21}, etc. Now V transforms lines into conics (ϕ'_2) through $O'_{11}\,O'_{12}\,O'_{13}$ and these are transformed by W into a net of quartics (ϕ''_4) with base points $(O''_{21}\,O''_{22}\,O''_{23})^2\ (O''_{11}\,O''_{12}\,O''_{13})^1$ (the index showing their order) which is the second homoloidal net of $T = VW$. By applying W^{-1}, V^{-1} successively we can obtain the first homoloidal net of T, $(\phi_4) \equiv (O_{11}\,O_{12}\,O_{13})^2\ (O_{21}\,O_{22}\,O_{23})^1$ of the same

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* *Rendiconti del Circolo Matematico de Palermo*, Vol 9 pp 256—259.

† Hudson; *loc. cit.*, p. 56.

‡ *Ibid.* p. 56, § 5.

nature as the other. If with this T_4 we take another T_2 with F-points in general position, we obtain a T_8 with three four-ple F-points, three double and three simple F-points. Thus by compounding μ T_2's with general F-systems we construct a transformation T_{2^μ} with the general characteristic $3^{2^{\mu-1}}, 3^{2^{\mu-2}} \dots 3^1$.

Next suppose that $O'_{11} \equiv O'_{21} = O'_1$ say. Then V transforms lines into conics (ϕ'_2) through $O'_1 O'_{12} O'_{13}$ of which one is an F-point for W and so on applying W a P-line will drop out and the second homoloidal net (ϕ''_3) will consist of cubics through $O''^2_1\, O''_{22}\, O''_{23}\, O''_{12}\, O''_{13}$. Taking another T_2 with one F-point at O''_1 and the other two in general position we get the Jonquiere's type for $n = 4$ and this can be generalised for all n.

5. De Jonquiere's Transformation with a Conic of fixed points.

Suppose the F-points of this transformation are $O_1^{n-1} (O_2 \dots O_{2n-1})^1$ and let the accented letters denote the F-points of the second system. If this transformation possesses a fixed conic, the conic must pass through F-points of total multiplicity $2n - 2$. Hence two cases are possible, either the $(2n - 2)$ simple F-points or any $(n - 1)$ simple F-points and the multiple F-point, should lie on the conic. This type of transformation with a conic of fixed points can be constructed by repeated application of T_2's

Let a pair of associated F-points $O_1 O'_1$ on the conic be given. Then apply Dickson's construction $(n - 1)$ times interchanging O_1, O'_1 and letting O_2, O'_2 vary on the conic and O_3, O'_3 on the plane. This gives a T_{jn} of the second type mentioned above. But when the multiple point $O \equiv O'$ is given in the plane, and not on the conic, T_{jn} can be obtained by a successive application of Dickson's construction with $O_1 O'_1$, $O_2 O'_2$ varying on the conic. This is of the first type and there exists transformations of both the types for all values of n.

6. Quartic Transformation with a Conic of fixed points.

There are two types of quartic transformations, the Jonquiere's type and the semi-symmetric type with characteristics $3^2.3$. This latter type of transformation can have a conic S for an invariant curve, when S passes either through the three double F-points or through two double and two simple F-points as mentioned before. Also there is an isolated

invariant point in the first case and none in the second. I will now consider the two cases separately and derive the general transformation equations and also consider the composition of such transformations as product of Dickson's T_2's.

6.1. *Type I.* Suppose that $O_1O_2O_3$ are the double F-points and $O_4O_5O_6$ the simple F-points of a semi-symmetric quartic transformation with a conic S of fixed points passing through $O_1O_2O_3$. The general transformation equations are

$$\rho x' = xf_3 + S.S_1$$
$$\rho y' = yf_3 + S.S_2$$
$$\rho z' = zf_3 + S.S_3.$$

Since f_3 and S intersect at the F.points on S, these equations take the form

$$\rho x' = x^2yz + S.S_1$$
$$\rho y' = xy^2z + S.S_2$$
$$\rho z' = xyz^2 + S.S_3.$$

Outside the conic S there is an isolated invariant point D. Hence the points of intersection of the cubics $xS_2 - yS_1$ and $xS_3 - zS_1$ other than the points of intersection of x and S_1 must be only the F-points and the invariant point D. So S_1, S_2, S_3 are conics through $O_1O_2O_3$ and their equations can be fixed by writing out the conditions for the three quartics $x^2yz + SS_1$, $xy^2z + SS_2$, $xyz^2 + S.S_3$ to pass through $O_4O_5O_6$.

6.2. A special case of this transformation has been studied by Dr. R. Vaidyanathaswamy* in which a fundamental point of the first order and one of the second order coalesce at each of $O_1O_2O_3$ so that these points correspond to line pairs. Also the points $O_1O_2O_3$ are F-points for the reverse transformation while the poles of the sides of $O_1O_2O_3$ with respect to the fixed conic S are F-points of the first order. The transformation is the result of performing successively (1) the operation P of taking the polar line in respect of the triangle $O_1O_2O_3$; (2) the operation C of taking the pole in respect of S, and

* Proc. Camb. Phil. Soc., Vol. 23, "(2.1) correspondences," p. 236.

(3) a certain quadratic involutory Cremona transformation I (Steinerian or generalised isogonal transformation) with the F-points $O_1O_2O_3$. This transformation has been resolved symmetrically into IVI where I is the Cremona involution with the centre of perspectivity V which carries $O_1O_2O_3$ into its polar triangle with respect to S for a fixed point.

6.3. *Composition of Type I.* The general quartic transformation with a conic of fixed points studied in (6.1) can be obtained as the product of three quadratic transformations. Suppose the three associate pairs of F-points O_1, O'_1; O_2, O'_2; O_3, O'_3 are given on the conic S Let U be the quadratic transformations with O_1, O'_1; O_2, O'_2 for associate pairs of F-points. Then the point of intersection of O'_1O_2 and $O_1O'_2$ is the third F-point $O_4 \equiv O'_4$. Consider next a similar transformation V with O'_1O_1; $O_3O'_3$ for associate pairs of F-points and $O_5 \equiv O'_5$ for the third F-point. Then UV defines a cubic transformation with $O_1{}^2O'_3O_5O'_2O_4''$ for F-points of the second system where O''_4 is the V-transform of O_4. If now we define a third transformation W with O'_2, O_2; $O'_3O_3O_6 \equiv O'_6$ for associate pairs of F-points and if O_4''' and O''_5 are the W-transform of O''_4 and O_5, then UVW is the quartic transformation with $O_1{}^2\, O_2{}^2\, O_3{}^2\, O_6\, O''_5\, O_4'''$ for the second F-system. The first F-system can be easily obtained by considering the product $W^{-1}\, V^{-1}\, U^{-1}$ and can be easily seen to be $O_1{}^2\, O_2{}^2\, O_3{}^2\, O_4\, O''_5\, O'''_6$ where O''_5 is the transform of O_5 in V^{-1} and O''_6 of O_6 in $V^{-1}\, U^{-1}$. We notice that the F-points of the second order for both systems are co-incident.

6.4. *Type II.* We will next consider the second type in which the conic S passes through $O_1O_2O_4O_5$ where O_1O_2 are double and O_4O_5 simple F-points. Then the transformation equations referred to $O_1O_2O_5$ can be put in the form

$$\begin{aligned}
\rho x' &= xz^2\,(y - z) + S\,.\,S_1 \equiv \phi_1\\
\rho y' &= yz^2\,(y - z) + S\,.\,S_2 \equiv \phi_2\\
\rho z' &= z^3\,\,(y - z) + S\,.\,S_3 \equiv \phi_3
\end{aligned}$$

where S_1, S_2, S_3 are quadratics in x, y, z and O_4 has co-ordinates (1,1,1). The freedom of variation for $S_1\ S_2\ S_3$ can be limited by considering the cubics $xS_2 - yS_1$, $xS_3 - zS_1$, to pass through $O_1\ O_2\ O_4\ O_5$ and the quartics ϕ_1, ϕ_2, ϕ_3 to pass through O_3, O_6.

6.5. ***Composition.*** We can also obtain this type of transformation by compounding three quadratic transformations with the same conic of fixed points. Suppose that we are given on the conic $O_1 O_2 O_4 O_5$ $O'_1\, O'_2\, O'_4\, O'_5$, such that $O_1\, O'_2$ and $O_2\, O'_1$ meet in O_6, and $O_1\, O'_5$, $O'_1\, O_5$, $O_2\, O'_4$, $O_4\, O'_2$ are concurrent at O_3, Consider the quadratic transformation U defined by the F-points $O_1 O_2 O_6$ and another V defined by the F-points O', O'_5, O_3 both having the conic S for a curve of fixed points. Also let O''_6 be the transform of O_6 in V. Define a third transformation W by the F-points $O'_2\, O'_4\, O_3$ and let O_6''' be the transform of O_6'' in this. Now UVW is the tramsformation considered in (6.4) with $O_1^2\, O_2^2\, O_3^2\, O_4\, O_5\, O_6'''$ for the second F-system. By considering the reverse transformation $W^{-1}\, V^{-1}\, U^{-1}$ it can be easily seen that the F-points of the first system are $O_1^2\, O_2^2\, O''^2_3\, O'_4\, O'_5\, O_6$. where O''_3 is the transform of O_3 in U^{-1}.

7. A Quartic transformation defined by a Conic.

Finally I will consider a general quartic transformation defined by a conic which arises as a natural sequence of the special case mentioned in (6.2). Given a triangle $\alpha\beta\gamma$ and a conic S, let $\alpha'\beta'\gamma'$ be the polar triangle of $\alpha\beta\gamma$ with respect to S and O the centre of perspectivity of $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$. Define V as the perspectivity carrying $\alpha\beta\gamma$ into $\alpha'\beta'\gamma'$ and I the quadratic Cremona involution with F-points at $\alpha\beta\gamma$ and O for a fixed point. Then the general quartic transformation is $\Gamma \equiv IVI$.

If referred to $\alpha\beta\gamma$

$$S \equiv ax^2 + by^2 + cz^2 + 2\,gzx + 2\,hxy + 2\,fyz = 0$$

$\alpha'\, \beta'\, \gamma'$ and O have co-ordinates (A, H, G), (H, B, F), (G, F, C) and (1/F, 1/G, 1/H) respectively where the capitals are the minors of the corresponding letters in the determinant :

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

The equations of transformation for the perspectivity are easily seen to be

$$\rho x' = Ax + Hy + Gz$$
$$\rho y' = Hx + By + Fz$$
$$\rho z' = Gx + Fy + Cz$$

and hence the equations of the quartic transformation T are:

$$\sigma x' = J_2 J_3$$
$$\sigma y' = J_3 J_1$$
$$\sigma z' = J_1 J_2$$

where
$$J_1 = Ayz + \frac{HF^2}{G^2} zx + \frac{GF^2}{H^2} xy$$
$$J_2 = \frac{HG^2}{F^2} yz + Bzx + \frac{FG^2}{H^2} xy$$
$$J_3 = \frac{GH^2}{F^2} yz + \frac{FH^2}{G^2} zx + Cxy.$$

$\alpha, \beta \gamma$ are F-points of the second order for this transformation and the simple F-points are

$$O' : \quad \left(\frac{1}{aF^2}, \frac{1}{hG^2}, \frac{1}{gH^2}\right)$$
$$O'' : \quad \left(\frac{1}{hF^2}, \frac{1}{bG^2}, \frac{1}{fH^2}\right)$$
$$O''' : \quad \left(\frac{1}{gF^2}, \frac{1}{fG^2}, \frac{1}{cH^2}\right).$$

A point P will be a fixed point of the transformation when

$$IVI\,(P) = P$$
$$\therefore \quad VI\,(P) = I\,(P)$$

or
$$V\,(P') = P' \text{ where } P' = I\,(P).$$

But the perspectivity can have only three fixed points and so there are only three fixed points for this transformation. This transformation is of the type studied by Hudson.*

* *Loc. cit.* p. 118.

SOME INDEFINITE INTEGRALS INVOLVING THE FUNCTIONS OF THE PARABOLIC CYLINDER.

BY RAMA SHANKAR VARMA, M.SC.

The functions of the parabolic cylinder are defined as the solutions of the differential equation:*

$$\frac{d^2 y}{dz^2} + Zy = 0,$$

where Z is a quadratic function of z. This differential equation may, without loss of generality, be taken to be of the form† adopted by Whittaker:

$$\frac{d^2 y}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4} z^2)\, y = 0, \qquad \ldots \quad (1)$$

where n is any constant. The standard solution of (1) is denoted by $D_n(z)$ which satisfies two sets of recurrence formulae, *viz*,

$$D_{n+1}(z) - z D_n(z) + n D_{n-1}(z) = 0, \qquad (2)$$

and

$$\frac{d\, D_n(z)}{dz} + \tfrac{1}{2} z\, D_n(z) - n D_{n-1}(z) = 0, \qquad \ldots \quad (3)$$

The theory of the parabolic cylinder function has been developed from time to time by various writers. Hitherto however, no systematic treatment of the function seems to have been given which can be regarded as corresponding to the theory of indefinite integrals involving Bessel Functions, studied by Lommel,‡ Sonine § and recently by Vint.‖ The object of this paper is an attempt to supply this deficiency.

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* Weber: *Math. Ann.*, Vol. I (1869), pp. 1—36.

† *Proc. Lond. Math. Soc.*, Series I, Vol. XXXV (1903), pp. 417—27.

‡ See his papers, "Zur Theorie der Besselsche Functionen," published in *Math. Ann.*, Vols. II, III & IV.

§ *Math. Ann.*, Vol XVI (1880), pp. 30—33.

‖ Journal of the *Lond. Math. Soc.*, Vol II (1927) pp. 102—105.

Thus in § 1, I consider integrals of the type $\int f(z)\, D_n(z)\, dz$. § 2 deals with $\int f(z)\, D_m\{\phi(z)\}\, D_n\{\psi(z)\}\, dz$. In the subsequent articles I have studied certain particular cases of the general integral considered in § 2.

1. Let us consider the integral

$$\int f(z)\, D_n(z)\, dz,$$

which we may suppose to be of the form

$$A(z)\, D_n(z) + B(z)\, D_{n+1}(z),$$

where $A(z)$ and $B(z)$ are to be determined.

Differentiating and making use of the formulae (2) and (3), we have

$$f(z)\, D_n(z) = \left\{ \frac{dA(z)}{dz} + \tfrac{1}{2} zA(z) + (n+1)\, B(z) \right\} D_n(z)$$
$$+ \left\{ \frac{dB(z)}{dz} - \tfrac{1}{2} zB(z) - A(z) \right\} D_{n+1}(z)$$

In order that $A(z)$ and $B(z)$ may not depend on the parabolic cylinder function, we take

$$A(z) \equiv \frac{d\,B(z)}{dz} - \tfrac{1}{2} z\, B(z),$$

and then

$$f(z) = \frac{d^2\, B(z)}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4} z^2)\, B(z).$$

Hence it follows, by virtue of (2) and (3), that

$$\int \{ B''(z) + (n + \tfrac{1}{2} - \tfrac{1}{4} z^2)\, B(z) \}\, D_n(z)\, dz = B'(z)\, D_n(z) - B(z)\, D_n'(z), \quad \ldots \quad (4)$$

where accents denote differentiation with respect to z.

The following reduction formula is an obvious consequence of (4):

$$\int z^{m+2}\, D_n(z)\, dz - 2(2n+1) \int z^m D_n(z)\, dz - 4m(m-1) \int z^{m-2}\, D_n(z)\, dz$$
$$= 4z^{m-1} \{ z\, D_n'(z) - m\, D_n(z) \}.$$

If, however, we start with the integral

$$\int z^{n+1} f(z)\, D_n(z)\, dz = z^{n+1} \{ A(z)\, D_n(z) + B(z)\, D_{n+1}(z) \},$$

we find in a similar manner that :

$$\int z^{n+1}\Big[B''(z)+\frac{2(n+1)}{z}B'(z)$$

$$+\left\{\frac{n(n+1)}{z^2}+n+\frac{1}{2}-\frac{1}{4}z^2\right\}B(z)\Big]D_n(z)\,dz$$

$$=[(n+1)z^n B(z)-\tfrac{1}{2}z^{n+2}B(z)+z^{n+1}B'(z)]\,D_n(z)$$

$$+z^{n+1}B(z)D_{n+1}(z) \qquad \ldots \quad (5)$$

2. Suppose now that

$$\int f(z)D_m\{\phi(z)\}D_n\{\psi(z)\}\,dz$$

$$=A(z)D_m\{\phi(z)\}D_n\{\psi(z)\}$$

$$+B(z)D_{m+1}\{\phi(z)\}D_n\{\psi(z)\}$$

$$+C(z)D_m\{\phi(z)\}D_{n+1}\{\psi(z)\}$$

$$+D(z)D_{m+1}\{\phi(z)\}D_{n+1}\{\psi(z)\} \qquad \ldots \quad (6)$$

On differentiating, we obtain, in the manner indicated in § 1, the following equations for the determination of A, B, C and D.

$$\left.\begin{aligned} f(z)&=A'+\tfrac{1}{2}(\phi\phi'+\psi\psi')A+(m+1)\phi'B+(n+1)\psi'C\\ 0&=B'-\phi'A-\tfrac{1}{2}(\phi\phi'-\psi\psi')B+(m+1)\psi'D\\ 0&=C'-\psi'A+\tfrac{1}{2}(\phi\phi'-\psi\psi')C+(m+1)\phi'D\\ 0&=D'-\psi'B-\phi'C-\tfrac{1}{2}(\phi\phi'+\psi\psi')D \end{aligned}\right\}\ldots \quad (7)$$

If h, a function of z, is defined by

$$2\frac{dh}{dz}-h\left(\frac{\phi''}{\phi'}+\frac{\psi''}{\psi'}\right)=\left(\frac{\phi''}{\phi'}-\frac{\psi''}{\psi'}\right)D'+$$

$$D\,[\tfrac{1}{2}(\phi\phi')^2-\tfrac{1}{2}(\psi\psi')^2-(\phi'^2-\psi'^2)-2(m\phi'^2-n\psi'^2)].$$

then we obtain from (7)

$$2\phi'C=D'-\phi\phi'D+h,$$

$$2\psi'B=D'-\psi\psi'D-h,$$

$$4\phi'\psi'A=hP(z)+2D''-D'Q(z)+DR(z),$$

and $f(z) = A' + \frac{1}{2}(\phi\phi' + \psi\psi')A + \frac{1}{2}\left(\overline{m+1}\,\frac{\phi'}{\psi'} + \overline{n+1}\,\frac{\psi'}{\phi'}\right)D'$

$$-\tfrac{1}{2}(\overline{m+1}\,\psi\phi' + \overline{n+1}\,\psi'\phi)D - \tfrac{1}{2}\left(\overline{m+1}\,\frac{\phi'}{\psi'} - \overline{n+1}\,\frac{\psi'}{\phi'}\right)h,$$

where
$$P(z) = (\phi\phi' - \psi\psi') - \left(\frac{\phi''}{\phi'} - \frac{\psi''}{\psi'}\right),$$

$$Q(z) = (\phi\phi' + \psi\psi') + \left(\frac{\phi''}{\phi'} + \frac{\psi''}{\psi'}\right),$$

and
$$R(z) = 2(m\phi'^2 + n\psi'^2) + \phi'^2 + \psi'^2 + \phi\phi'\psi\psi' - \tfrac{1}{2}(\phi\phi')^2 - \tfrac{1}{2}(\psi\psi')^2.$$

3. As a special case, let us take $\phi(z)$ and $\psi(z)$ to be multiples of z, say, kz and lz respectively. Then the condition that (6) may be true, gives

$$2\,kC = D' - k^2 zD + h,$$
$$2\,lB = D' - l^2 zD - h,$$
$$4\,klA = h\,P(z) + 2D'' - Q(z)\,D' + R(z)\,D,$$

and
$$f(z) = A' + \tfrac{1}{2}(k^2 + l^2)\,zA + \tfrac{1}{2}\left(\overline{m+1}\,\frac{k}{l} + \overline{n+1}\,\frac{l}{k}\right)D'$$
$$-\tfrac{1}{2}\,zkl\,(m + n + 2)\,D - \tfrac{1}{2}\left(\overline{m+1}\,\frac{k}{l} - \overline{n+1}\,\frac{l}{k}\right)h$$

where
$$2\,\frac{dh}{dz} = \left[\tfrac{1}{2}z^2(k^4 - l^4) - (k^2 - l^2) - 2(mk^2 - nl^2)\right]D,$$
$$P(z) = z(k^2 - l^2),$$
$$Q(z) = z(k^2 + l^2),$$
and
$$R(z) = 2(mk^2 + nl^2) + k^2 + l^2 - \tfrac{1}{2}z^2(k^4 + l^4) + k^2l^2z^2.$$

Two interesting cases arise.

Case I. $k = l$.

Here if we suppose that $D = z^{\lambda+3}$, the indefinite integral (6) reduces to

$$\int \{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)z^{\lambda} + 2(\lambda+3)(\lambda+4)(m+n+1)z^{\lambda+2}$$
$$+ (\overline{m-n}^2 - \overline{\lambda-4}^2)\,z^{\lambda+4}\}\,D_m(z)\,D_n(z)\,dz$$

$$= (\lambda+4)\{(\lambda+2)(\lambda+3)z^{\lambda+1} - (\lambda-m-n+2)z^{\lambda+3}\} D_m(z) D_n(z)$$
$$+ \{(\lambda+3)(\lambda+4)z^{\lambda+2} - (\lambda-m+n+4)z^{\lambda+4}\} D_{m+1}(z) D_n(z)$$
$$+ \{(\lambda+3)(\lambda+4)z^{\lambda+2} - (\lambda+m-n+4)z^{\lambda+4}\} D_m(z) D_{n+1}(z)$$
$$+ 2(\lambda+4)z^{\lambda+3} D_{m+1}(z) D_{n+1}(z) \qquad \ldots \quad (8)$$

It is easy to show that (8) leads to the evaluation of

$$\int z_s D_m(z) D_n(z)\, dz \ (m = n), \quad s \neq \text{an even integer}$$

in the form

$$[AD_m(z) + BD_{m+1}(z)]\, D_n(z) + [CD_m(z) + DD_{m+1}(z)]\, D_{n+1}(z),$$

where A, B, C and D are polynomials.

Case II. $k = 1, \ l = i.$

Putting $D = z^{\lambda+3}$, we obtain from (6)

$$\int \{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)z^{\lambda} + 2(\lambda+3)(\lambda+4)(m-n)z^{\lambda+2}$$
$$+ (m+n+\lambda+5)(m+n-\lambda-3)z^{\lambda+4}\} D_m(z) D_n(iz)\, dz$$
$$= \{(\lambda+2)(\lambda+3)(\lambda+4)z^{\lambda+1} + (m-n)(\lambda+4)z^{\lambda+3}$$
$$- (m+n+\lambda+5)z^{\lambda+5}\} D_m(z) D_n(iz)$$
$$+ \{(\lambda+3)(\lambda+4)z^{\lambda+2} + (m+n+\lambda+5)z^{\lambda+4}\} D_{m+1}(z) D_n(iz)$$
$$+ i\{(\lambda+3)(\lambda+4)z^{\lambda+2} - (m+n+\lambda+5)z^{\lambda+4}\} D_m(z) D_{n+1}(iz)$$
$$+ 2i(\lambda+4)z^{\lambda+3} D_{m+1}(z) D_{n+1}(iz) \qquad \ldots \quad (9)$$

As particular cases of (9) we have when $\lambda = -4$,

$$(m+n+1)^2 \int D_m(z) D_n(iz)\, dz = -z D_m(z) D_n(iz)$$
$$+ D_{m+1}(z) D_n(iz) - i D_m(z) D_{n+1}(iz), \qquad \ldots \quad (10)$$

and for $\lambda = -3$

$$(m+n)(m+n+2) \int z D_m(z) D_n(iz)\, dz$$
$$= \{(m-n) - (m+n+2)z^2\} D_m(z) D_n(iz)$$
$$+ (m+n+2) z D_{m+1}(z) D_n(iz)$$
$$- i(m+n+2) z D_m(z) D_{n+1}(iz)$$
$$+ 2i D_{m+1}(z) D_{n+1}(iz) \qquad \ldots \quad (11)$$

In this case we conclude that integrals of the type

$$\int z^s D_m(z) D_n(iz)\, dz$$

may be expressed in the form

$$[A D_m(z) + B D_{m+1}(z)]\, D_n(iz) + [C D_m(z) + D D_{m+1}(iz)]\, D_{n+1}(iz),$$

where A, B, C and D are polynomials.

4. Next if we put $B = C = 0$ in the 2nd and 3rd expressions of (7), we have

$$A^2 = (m+1)(n+1) D^2, \quad \phi'^2 = \frac{n+1}{m+1} \psi'^2.$$

Hence

$$A = \pm \sqrt{(m+1)(n+1)}\, D, \quad \phi(z) = a \pm \sqrt{\frac{n+1}{m+1}}\, \psi(z),$$

where a is any constant.

Putting $\psi(z) = z$, we conclude from (6) that

$$\int \left[A' + \frac{1}{2}\left\{ z \pm \sqrt{\frac{n+1}{m+1}} \left(a \pm \sqrt{\frac{n+1}{m+1}}\, z \right) \right\} A\right] \times$$

$$D_m\left\{ a \pm \sqrt{\frac{n+1}{m+1}}\, z \right\} D_n(z)\, dz$$

$$= A D_m\left\{ a \pm \sqrt{\frac{n+1}{m+1}}\, z \right\} D_n(z)$$

$$\pm \frac{1}{\sqrt{(m+1)(n+1)}}\, A D_{m+1}\left\{ a \pm \sqrt{\frac{n+1}{m+1}}\, z \right\} D_{n+1}\, z.$$

In the particular case of A being a constant, we find

$$\frac{1}{2} \int \left\{ z \pm \sqrt{\frac{n+1}{m+1}} \left(a \pm \sqrt{\frac{n+1}{m+1}}\, z \right) \right\} \times$$

$$D_m\left\{ a \pm \sqrt{\frac{n+1}{m+1}}\, z \right\} D\, (z)\, dz$$

$$= D_m\left\{ a \pm \sqrt{\frac{n+1}{m+1}}\, z \right\} D_n(z)$$

$$\pm \frac{1}{\sqrt{(m+1)(n+1)}}\, D_m\left\{ a \pm \sqrt{\frac{n+1}{m+1}}\, z \right\} D_{n+1}(z)$$

Putting $m = n$, we obtain

$$\tfrac{1}{2}\int \{z \pm (a \pm z)\}\, D_m(a \pm z)\, D_m(z)\, dz$$

$$= D_m(a \pm z)\, D_m(z) \pm \frac{1}{m+1} D_{m+1}(a \pm z)\, D_{m+1}(z).$$

5. Finally we shall obtain some interesting results from the integrals discussed in § 3.

If in (10) we put $n = -m - 1$, we have

$$m\, D_{-m-1}(iz)\, D_{m-1}(z) + i\, D_m(z)\, D_{-m}(iz) = \text{a constant, (say } \alpha).$$

In order to find the value of α, we put $z = 0$ in the expression on the left side. We find then

$$m\, D_{-m-1}(iz)\, D_{m-1}(z) + i\, D_m(z)\, D_{-m}(iz) = \sin\tfrac{1}{2}m\pi + i\cos\tfrac{1}{2}m\pi,$$

a result due to Dr. Gorak Prasad.*

If in (11), we put $n = -m$, we obtain in a similar manner

$$(m - z^2)\, D_m(z)\, D_{-m}(iz) + z\, D_{m+1}(z)\, D_{-m}(iz) - iz\, D_m(z)\, D_{-m+1}(z)$$

$$+ i\, D_{m+1}(z)\, D_{-m+1}(iz) = \frac{m}{2}\left[2\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right].$$

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* *Proc. Benares Math. Soc.*, Vol. IV (1922), pp. 21—25.

ON MR. RAMASWAMY AIYAR'S TOPOLOGICAL PROBLEM.

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In view of the interest which Mr. Ramaswamy Aiyar's prize problem has aroused and the several proofs of it appearing in Part II of this *Journal*, it may be of value to point out the order of ideas in geometry to which it is related and, while discussing it in detail, draw attention to problems in other fields which it suggests.

The problem belongs to a compartively recent branch of mathematics known as *Topologie* or *Analysis situs* which may be said to date from Listing's *Vorstudien zur Topologie* (1847). One of the earliest questions in this field to receive attention was Euler's Konigsberg Bridge Problem, whether the seven bridges over the Pregel could be traversed in succession without passing more than once over the same bridge. Among other interesting questions in this field which have received attention are the problem of knots, the eight queens problem, knight's path on a chessboard, labyrynths, and the famous four colour problem.

Klein found in group theory the basis for the classification of different types of geometrical problems, and the ideas enunciated by him in his Erlanger Programm (1872) have proved a fruitful point of departure for many important investigations extending over half a centnry. According to this view, a geometry is the study of those properties of a configuration which are unaltered by the operations of a transformation group. Thus suppose we accept the standpoint of analytical geometry and regard a point as defined by the mutual ratios of three ordered numbers x_1, x_2, x_3, and a straight line as the totality of "points" satisfying a linear equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$. If we subject these operands (points and aggregates of points) to arbitrary linear transformations

$$x'_r = a_{r1}x_1 + a_{r2}x_2 + a_{r3}x_3 \qquad (r = 1, 2, 3),$$

which form the general linear group, we have "projective analytic geometry" which studies properties of figures unaltered by all these transformations, such as the degree of a curve which is a numerical invariant, or the incidence relation of a point and a line which is a two-valued invariant relation. If instead of the whole group of linear transformations we take a sub-group thereof, we have a more specialised geometry with a larger number of invariants and therefore richer in content. For example, when the sub-group consists of those transformations which carry two fixed points into themselves, we have a geometry which is essentially identical with the ordinary Euclidean geometry of lines and angles. On the other hand we may consider transformation groups containing the linear group as a sub-group such as the totality of real continuous (1,1) transformations of a space manifold; and the study of the properties of aggregates of elements which are invariant for these constitute the subject matter of *Analysis situs.* In this order of ideas, the distinction between a straight line and a curve vanishes, and the degree of a curve is a meaningless term. The number of elements of each type (points, curves, etc.) are obvious numerical invariants, while incidence, succession, etc. are properties which persist and have a place in this geometry.

The introduction of a carrying analytic space-framework though necessitated by the ideas of the Erlanger Programm in order to provide the operands of the transformation group, is not indispensable for formulating Mr. Ramaswamy Aiyar's problem which may be stated in the abstract as follows:

> There are n entities any two of which are capable of a relation which is symmetrical but not transitive, called "direct connection." It is required to find the least number of direct connections to ensure that any two of the entities shall be related by a chain of direct connections, and to determine the number and the structure of all possible modes of direct connection which bring this about.

As an example, let the given entities be n letters, and let "direct connection" be interpreted as the transposition which interchanges them. The problem is then one in the theory of substitution groups, and seeks the minimum number of transpositions which generate a transitive substitu-

tion* group in n letters, and the several possible sets of generating transpositions. A particular case was proposed by Mr. Ramaswamy Aiyar as a prize problem in this *Journal* (Vol. 18 Part II, page 97, question 1547) in the following terms :—

> Show that the number of ways of connecting n villages with one another by $n-1$ roads, each of them connecting two villages directly is n^{n-2}.

The notation and nomenclature of this paper relate to this problem.

Such point manifolds have been studied under the name of Streckenkomplexe, Linensystem, Reseaux, Graphes, by several writers† and the *Memorial des Sciences Mathematiques* devotes a separate fascicule (No. XVIII) to this subject. The particular types of Reseaux which come under consideration here are those in which there are no rings (*vide* 1.2) and have been studied under the name "trees," Fr. "arbres," Ger. "Baume." Here again there are two types of problems according as we make each road an one-way path or permit its being traversed in either direction. For the former case, $n-1$ roads are both necessary and sufficient; for the latter there must exist at least one ring — the least number of roads necessary in this case is n—while there are modes of connection with more than n roads where the removal of any one will prevent a passage from any village to any other. The subject has been investigated by Listing, Cayley, Chuard, MacMahon, Sylvester, Tebay and others (*vide* bibliography at the end of the fascicule referred to above).

In this paper I confine myself to the study of Mr. Ramaswamy Aiyar's problem with non-oriented paths. Sections 1 and 2 explain the notation and prove certain preliminary propositions useful in the discussion of the enumerative problem.

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* It is easy to show that a transitive group whose generating operations are all transpositions is neceessarily the symmetric group of proper degree, *vide* Section 7).

† Vide *Encyclopadie der Mathematischen Wissenschaften* III (i) Analysis Situs, pages 171—178. The following books may also be consulted with advantage; Ball: *Mathematical Recreations and Essays*; Ahrens: *Mathematische Uhterhaltungen und Spiele; Cambridge Mathematical Colloquium*; Sainte Lague: *Les Reseaux (ou Graphes)* [Mem des Sciences Maths No. 18]. In the absence of the necessary papers for consultation I am unable to say whether Mr. Ramaswamy Aiyar's result is known.

In the next section various recurrence relations between $F(n)$, the number of ways of connecting n villages by $n-1$ roads and $F(n-1)$, $F(n-2)$, etc. are obtained. Three of these, namely, (3.1), (3.3), and (3.6) furnish three different proofs by induction of Mr. Ramaswami Aiyar's formula, $F(n) = n^{n-2}$, while the fourth (3.4) expresses $F(n)$ in terms of certain functions $\phi(n, t)$ and provides a result of some interest (5.8), which is probably new. Section 6 deals with the nomenclature of the several possible solutions, and furnishes a direct proof of (5.4).

The last section is a search for equivalent problems. Two of these are pointed out, one of which relates, as pointed out above, to the theory of permutation groups, and the other to a problem in matrix algebra.

The latter takes the very striking form of finding all the non-equivalent uniaxial $(n-1)$th roots of a matrix I all of whose elements are positive and non-zero, two matrices being "equivalent" if their zero and non-zero elements correspond, the laws of addition and multiplication being the same as in ordinary matrix theory. The study of matrix algebra with Boolean elements appears to be a fresh field offering many interesting problems for investigation. Thus it will be seen from the last section that the symmetric uniaxial solutions of the matrix equation $X^r = I$ give the several modes of connecting n villages so that there is a passage from any village to any other involving the use of not more than r roads, while the zero axial solutions furnish modes of connection in which there is always a path between any two villages involving exactly r roads, retraversing the same road being however permitted,

1. Notation, Definitions,

I denote the several villages by the symbols $v_1, v_2 \dots v_n$, and sometimes, when there is no ambiguity, merely by their suffixes. A road connecting v_r and v_s directly is denoted by (rs), and its existence asserted by the relation $(rs) = 1$, while $(rs) = 0$ denies it. We write $(rstu \dots)$ for $(rs) . (st) . (tu) . (\dots) \dots$, etc., so that $(rstu) = 1$, implies $(rs) = 1$ $(st) = 1$, etc. ... (1.1)

If $(rst \dots r) = 1$, where all the symbols are different except the the first and the last which are identical, the villages r, s, t, etc., are said to form a *ring*.

A number of villages are said to form a "connected system" or simply a "system" if between any two villages v_r and v_s of the system there exists a relation of the form $(rpq \ldots s) = 1$, wher $v_p, v_q \ldots$ are villages belonging to the system.

Two or more systems are said to be "unconnected" or "isolated" if the villages of each system are connected, but there is no route* connecting a village of one system to a village of any other system.

A system of villages are said to be "simply connected," if they form a connected system, such that the destruction of any one of the directly connecting roads* will render the system disconnected. ... (1.5)

A village which is directly connected with only one other village is a "*terminus*." All others are *junctions* and the *order* of a junction is the number of villages with which it is directly connected. These latter are its *neighbours*; thus a terminus has only a single neighbour. ... (1.6)

$f(n, t)$ = the number of different ways of simply connecting n villages so as to have t assigned villages and these only for terminii;

$F(n)$ = the total number of ways of simply connecting n-villages.

2. Preliminary Propositions.

The following preliminary propositions are easily proved;

r isolated systems of villages can be connected so as to form a single connected system, by means of $r - 1$ direct connections, but not by a smaller number.

Corollary. The number of roads in a simply connected system of n villages is $n - 1$. In a simply connected system if one road is destroyed, the villages are divided into two isolated systems; and generally, if $r - 1$ roads are destroyed, the villages form r isolated systems ... (2.2)

---

* Throughout this paper the word "road" is used synonymously with "direct connection." Where two villages are connected through a chain of connections, the chain is referred to by the word "route."

If in a connected system of villages, there exists a ring then the connection is not simple * ... (2.3)

In a connected system there are routes from any village to any other Let $(p \ldots x \ldots x \ldots q) = 1$, be one possible route from v_p to v_q. If any village x be passsed twice over during the journey as in the above, we may modify the route by omitting all the letters between the two x's and also one of the letters x. The modified route does not involve any new direct connections and is thus a possible path. When this has been done for every letter which occurs twice, the resulting route is called a "standard" route. In a connected system there are thus standard routes from any village to any other. It may be shown that:

In a simply connected system, there is one and only one standard route between any two villages ; ... (2.4)

for it is easy to prove that the existance of more than one standard route implies also the existance of a ring.

Every simply connected system has at least two terminii ... (2.5)

This is obvious for $n = 2$. Suppose n exceeds 2. Then, if there are not two terminii, there must be at least one junction. From it proceed two roads connecting it directly with two other villages. Each of these is either a terminus or is connected directly with at least one other village and so on As the number of villages is finite, this process can terminate either by each route ending in a terminus, or containing a ring, or the two routes meeting again. The last two hypotheses are impossible for a simply connected system by (2.3) and (2.4) respectively. There are, therefore, at least two terminii.

If a terminal village and the road leading from it to its neighbour be destroyed, the remaining villages and roads form a simply connected system. ... (2.6)

---

* The words "simply connected" as used here, may be interpreted in the Reimannian sense, for if the roads be regarded as strips having breadth and the villages as simply connected regions, the villages and roads form a simply connected region since owing to the absence of "rings" there are no non-vanishing circuits. This may also be inferred from (2.6) for the transformation referred to there is one which does not alter the connectivity of the whole region.

for in a connected system there is at least one standard route from any village to any other, and no standard route can have a terminus for an intermediate village since, if it did, it would have to pass twice through the neighbouring village of the terminus. Hence the destruction of a terminus and the village leading to it cannot affect the connectivity of the others. It may be shown that if the residual system is not simply connected, the original system cannot have been simply connected.

3. Methods of Enumeration.

In this section are obtained certain recurrence formulae for $F(n)$ which follow from the preceeding propositions.

Consider first all simple connections of n villages *which involve a particular direct connection* say (12). Were this road destroyed, the n villages would, by (2 2) be separated into 2 isolated systems of say r and $n-r$ villages of which one will contain v_1 and the other v_2. The number of ways of selecting $r-1$ from among the $n-2$ villages other than v_1 and v_2 is $(n-2)!/(r-1)!\,(n-r-1)!$ and when this has been done, there are $F(r)$ ways of connecting the $r-1$ village with v_1 and $F(n-r)$ different ways of forming a simply connected system out of v_2 and the $n-r-1$ other villages. Hence the total number of ways of connecting n villages simply all of which involve a particular road (12) is ${}_{n-2}C_{r-1}\ F(r)\ F(n-r)$. The same argument applies to any of the ${}_nC_2$ roads (rs), and when these have all been taken into account every possible method of simply connecting n villages will have been considered. But in this reckoning each method will be counted $(n-1)$ times, once corresponding to each of the $n-1$ roads it contains. Hence we have:

$$\sum_{r=0}^{n=2} {}_{n-2}C_r\ F(r+1)\ F(n-r-1) = F(n)\,.\,(n-1)/{}_nC_2$$
$$= 2\,F(n)/n. \qquad \ldots \quad (3.1)$$

Again, let us count all possible methods of simply connecting n villages *which have a particular village*, say v_1 *for a terminus.* These may all be obtained by connecting the other $n-1$ villages simply, and selecting one of them to be directly connected with v_1; hence the required number is $(n-1).F(n-1)$. Consider next all possible methods of connection in which v_1 is a junction of the second order. To enumerate all these, we first divide the $n-1$ villages other than v_1 into two groups of say n_1+1 and n_2+1 villages $(n_1+n_2=n-3)$,

connect simply the villages in each group, and select a particular village from each to be connected directly with v_1. Thus the required number is

$$\frac{1}{2}\sum\frac{(n-1)!\,(n_1+1).(n_2+1)}{(n_1+1)!\,(n_2+1)!}\,F(n_1+1).F(n_2+1).$$

the summation extended to all values of n_1 and n_2 for which

$$n_1+n_2=n-3,$$

the factor $\frac{1}{2}$ being due to each method of connection being counted twice in this computation, by the interchange of n_1 and n_2. This does not happen when $n_1=n_2$, but in this case, the number of ways of dividing $n-1$ villages into two equal groups is

$$\frac{1}{2}\,\frac{(n-1)!}{(n_1+1)!\,(n_2+1)!}$$

so that the above formulae applies to this case also.

Similarly, it is easy to show that the number of possible modes of connection in which v_1 is a junction of order r is

$$\frac{1}{r!}\sum\sum\cdots\sum\frac{(n-1)!}{n_1!\,n_2!\,\ldots\,n_r!}\,F(n+1),\,F(n_2+1)\ldots F(n_r+1) \qquad \ldots \quad (3.2)$$

where $n_1 n_2 \ldots n_r$ take all possible sets of values for which

$$n_1+n_2+\ldots n_r=n-r-1.$$

Since v_1 must in every mode of connection be a terminus or a junction of some order, the sum of all these must be $F(n)$, the total number of possible methods of connecting the n villages simply. Hence :*

$$F(n)=(n-1)\,F(n-1)+\tfrac{1}{2}\sum\frac{(n-1)!}{n_1!\,n_2!}\,F(n_1+1)\,F(n_2+1)$$

$$+\frac{(n-1)!}{3!}\sum\sum\ldots\ldots\text{etc.} \qquad \ldots \quad (3.3)$$

---

* The same method, I find, has been also used Prof. G N. Watson in his Solution to Q 1547 p *194—197*. The apparent differences between the formula (3.2) and the one given at the begining of page *196* are due to greater freedom of the ordered set $n_1 n_2 \ldots n_r$ of which two or more may be equal. If a of the n's have the value r, b of them the value s, etc., the term in (3.2) will be repeated $r!/a!\,b!\ldots$ times and their sum is identical with the expression given by Prof. Watson except that n is replaced by $n+1$ as it should be.

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Notes and Questions.

On the Prize Problem (Q. 1547).

Taking the villages in any order, denote them by the symbols $v_1, v_2, \dots v_n$ which we shall speak of as letters. Let us call $v_1, v_2, \dots v_n$ the fundamental permutation, or permutation F.

We shall show that, corresponding to any system, R, of $n - 1$ roads connecting the villages, we can, by process I below, derive a permutation, $J \equiv y_1 y_2 \dots y_{n-2}$, where $y_1, y_2 \dots y_{n-2}$ each denote one of the letters $v_1, v_2, \dots v_n$ and need not be necessarily distinct from one another.

We shall also show that if any permutation J, as above described be first given, then, by process II below, we can derive a system, R, of $n - 1$ roads connecting the villages.

It will then be observed that these two processes are reciprocal to one another That is, if a road system R leads to a permutation J under process I, the permutation J will, under process II, lead us back to the road system R; and *vice versa.*

When we have proved so, it will follow that the number of possible road systems, which is in question, is equal to the number of possible permutations J—which is evidently equal to n^{n-2}.

In proving this we shall also find that, along with each permutation $J \equiv y_1 y_2 \dots y_{n-2}$, there goes a permutation $T \equiv x_1 x_2 \dots x_{n-2}\, x_{n-1}\, x_n$, in which the x's each stand for one of the letters $v_1, v_2, \dots v_n$ (without repetitions) such that the road system, R, associated with it is

$$(x_1 y_1),\ (x_2 y_2),\ \dots\ (x_{n-2}\, y_{n-2}),\ (x_{n-1}\, x_n).$$

PROCESS I.

2. When a system, R, of $n - 1$ roads connecting the n villages is first given, let us have a plan P showing the villages and the roads.

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By a 'junction' let us mean a village at which two or more roads meet; and by a 'terminal' a village which is not a junction. In a plan like P (involving more than two villages) it will be obvious that no set of the roads will form a circuit. There will be at least two terminals. We shall have other readily recognized properties, for example, a village with which a terminal is connected directly must be a junction; and so on. We proceed by steps as follows:

(1) Let the first village of the permutation F that is a terminal in plan P be denoted by x_1: and let the village with which it connects directly be denoted by y_1. Let the village x_1 and the road x_1y_1 (but not the village y_1) be *erased* from the plan P: and let us call what is left of the plan P as plan P_1. [We may merely mark a zero across the road x_1y_1 to indicate the erasure of the terminal x_1 and the road x_1y_1.]

(2) Let the first village of the permutation F (with x_1 struck off from it if you please) that is a terminal in plan P_1 be denoted by x_2; and let the village with which it connects directly be denoted by y_2. Then let the village x_2 and the road x_2y_2 be erased from plan P_1; and let us call what is left of it as plan P_2.

(3) Let the first village of the permutation F (with x_1, x_2 struck off from it if you please) that is a terminal in plan P_2 be denoted by x_3 and the village with which it is connected directly by y_3. Then, by erasing x_3 and the road x_3y_3 we get the plan P_3.

Proceeding in this way ($n - 2$ times in all) we get the villages denoted by $x_1, y_1, x_2, y_2, \ldots x_{n-2}, y_{n-2}$ successively defined. And we are left with a plan P_{n-2} containing 2 villages only and the road joining them. Let us denote them by x_{n-1}, x_n — the village with the lower suffix in v being taken as x_{n-1} [Then it will be found that x_n is v_n exactly].

As they are found, let us set the x's in a line, forming the permutation $T \equiv x_1x_2 \ldots x_{n-2}\, x_{n-1}\, x_n$; and similarly we set the y's in a line, forming the permutation $J \equiv y_1y_2 \ldots y_{n-2}$; and we take J as the permutation corresponding to the given road system R. We note that, as between T and J, the roads of the given system are yielded as

$$(x_1y_1),\ (x_2y_2),\ \ldots\ (x_{n-2}\ y_{n-2}),\ (x_{n-1}\ x_n).$$

PROCESS II.

3. We next proceed to show how, if any permutation $J \equiv y_1 y_2 \dots y_{n-2}$ be first given we can fix a system of $n - 1$ roads, R, connecting the villages as corresponding to the permutation. We proceed by steps as follows:

(1) Let the first village of the permutation F that does not occur in J be denoted by x_1. Set x_1 under y_1 in J; and denote by J_1 the permutation $(x_1 y_2 y_3 \dots y_{n-2})$.

(2) Let the first village of the permutation F that does not occur in J_1 be denoted by x_2. Set x_2 under y_2 in J; and denote by J_2 the permutation $(x_1 x_2 y_3 y_4 \dots y_{n-2})$.

(3) Let the first village of the permutation F that does not occur in J_2 be denoted by x_3. Set x_3 under y_3 in J; and denote by J_3 the permutation $(x_1 x_2 x_3 y_4 \dots y_{n-2})$. Proceeding like this ($n$ — 2 times in all) we shall get the villages $x_1, x_2 \dots x_{n-2}$ of the permutation F set respectively under $y_1, y_2, \dots y_{n-2}$ in the given permutation J. The two remaining villages of the permutation F will now be denoted by x_{n-1} and x_n, the village with the lower suffix in v being denoted by x_{n-1}. [It will be found that x_n will be v_n exactly].

The road system R corresponding to the given permutation J is now defined to consist of the $n-1$ roads

$$(x_1 y_1),\ (x_2 y_2),\ \dots (x_{n-2}\ y_{n-2}) \text{ and } (x_{n-1}\ x_n).$$

A point here will be to show that these $n - 1$ roads really connect all the villages. For this purpose we observe as follows: The definition of x_2 shows that it is different from x_1. The definition of x_3 shows that it is different from x_1, x_2; and so on. Hence all the x's are different from one another. They are simply $v_1, v_2, \dots v_n$ written in some order. Again from the definitions of $x_1, x_2, \dots x_p$ we see that each of these is different from y_p. Since y_p is contained in the total set (x) we should have y_p in the set $x_{p+1}, x_{p+2}, \dots x_n$; that is y_p is the same as x_q for some value of $q > p$. Hence it can be inferred that starting from any x_p and going along roads of the system $(x_p y_p)$, $(x_q y_q)$, ... we can always reach x_n; which shows that there is connection between all the villages.

4. It remains to show that if from a road system, R, we derive a permutation J by process I, then, by application of process II to this J we get back our road system R; and *vice versa*. The reader will be able to verify this by working out some actual instances with, say, $n = 8$ or 10. The verification will suggest to him the simple explanation, or reasoning, as to how the processes I and II are really reciprocal to one another.

Corollary 1.

5. Let n villages be connected by $n - 1$ roads. A junction at which k roads meet may be said to be of order $k-1$. In the corresponding permutation J it is obvious that every y is a junction and every junction occurs among the y's as often as its order. It follows that the sum of the orders of all the junctions is equal to $n - 2$.

Let it be prescribed that certain specified villages $\eta_1, \eta_2, \dots \eta_r$ are to be junctions of assigned orders $\lambda_1, \lambda_2 \dots \lambda_r$ where $\Sigma\lambda = n - 2$. The number of ways possible of connecting the villages by $n - 1$ roads subject to these prescriptions is obviously the number of possible permutations J of $n - 2$ letters of which λ_1 are η_1, λ_2 are η_2, and so on. Hence the number of ways is

$$(n - 2)! / (\lambda_1)! . (\lambda_2)! \dots (\lambda_r)!$$

[This is Mr. A. A. Krishnaswami Ayyangar's main theorem in his solution of my problem.]

Corollary 2.

6. Let there be n groups of villages $G_1, G_2 \dots G_n$ containing respectively $a_1, a_2, \dots a_n$ villages. The groups being unconnected with one another by roads, let it be required to find in how many ways the groups may be all connected with one another by $n - 1$ roads, the mode of connecting two groups being by laying a road from any village of the one to any village of the other.

Answer: Treating the groups as single villages, if $n - 1$ roads connect them, these can be represented, as per method given above, by the help of two permutations

$$J \equiv Y_1, Y_2, \dots Y_{n-2}$$

and

$$T \equiv X_1, X_2, \dots X_{n-2}, X_{n-1}, X_n$$

where the X's and Y's represent individual ones among the groups $G_1, G_2, \ldots G_n$, the representation of the roads being

$$(X_1Y_1), (X_2Y_2) \ldots (X_{n-2}\ Y_{n-2}), (X_{n-1}\ X_n).$$

Let $\zeta_1, \zeta_2, \ldots \zeta_n$ denote the number of villages in the groups $X_1, X_2, \ldots X_n$; similarly let $\eta_1, \eta_2, \ldots \eta_{n-2}$ be the number of villages in the groups $Y_1, Y_2 \ldots Y_{n-2}$. In the above representation of the $n - 1$ roads connecting the groups, (X_1Y_1) now stands, not for a single road, but for one of $\zeta_1\eta_1$ possible roads. The total number of systems of roads embraced in the particular representation

$$(X_1Y_1), (X_2Y_2), \ldots (X_{n-1}X_n)$$

accordingly is

$$\zeta_1\eta_1 \cdot \zeta_2\eta_2 \ldots \zeta_{n-2}\eta_{n-2} \cdot \zeta_{n-1}\zeta_n;$$

that is, it is equal to

$$\zeta_1\zeta_2 \ldots \zeta_n \times \eta_1\eta_2 \ldots \eta_{n-2};$$

that is, it is equal to

$$a_1a_2 \ldots a_n \times \eta_1\eta_2 \ldots \eta_{n-2}$$

where $\eta_1\eta_2 \ldots \eta_{n-2}$ is a certain permutation of $n - 2$ of the letters $a_1a_2, \ldots a_n$ (with repetitions allowed). Hence the aggregate number for all possible representations as above $= a_1a_2 \ldots a_n \Sigma\, \eta_1\eta_2 \ldots \eta_{n-2}$ where $\eta_1, \eta_2, \ldots \eta_n$ may, independently of one another, denote any of the quantities $a_1, a_2, \ldots a_n$. *The aggregate number is therefore*

$$= a_1a_2 \ldots a_n \times (a_1 + a_2 + \ldots a_n)^{n-2}.$$

[This is *not by the multinomial theorem*; but by a direct idea of what the $(n - 2)$th power of $(a_1 + a_2 + \ldots + a_s)$ means when viewed as the product of $n - 2$ separate factors. We thus prove the generalization of my result given by Mr. Kashikar; and later also by Mr. A. A. Krishnaswami Aiyangar].

Corollary 3.

This generalization furnishes a solution of a problem arising in connection with my original problem of connecting n villages by $n - 1$ roads. Supposing *certain* of these roads have been laid down to begin with, in how many ways can the rest of them be laid? [Both Kashikar and A. A. Krishnaswami Aiyangar have noticed this application].

Corollary 4.

Let there be $s+1$ 'groups,' the first consisting of r villages and the remaining s consisting of single villages. The number of ways of connecting the group of r villages with the s isolated villages by s roads is

$$r\,(r+s)^{s-1}.$$

[Mr. Kashikar has proved this in a very simple and interesting manner; and it forms an extension of my result, by itself].

Corollary 5.

From the last theorem of Kashikar, it is easy to show elementarily, that if x and y be positive integers and x_n denote $(x+n)^{n-1}$, then,

$$(x+y)_n = x_n + nC_1\, x_{n-1}y_1 + nC_2\, x_{n-2}y_2 + \ldots + nC_{n-1}\, x_1 y_{n-1} + y_n$$

The restriction that x and y are positive integers can now be removed: and we see that the above is an identity holding for all values of x and y.

This is a theorem, due to Prof. Sircom, that I saw in the *Educational Times* for January 1893. I was unable to prove it by Elementary Algebra, though I proved it by means of Lagrange's Theorem and the properties of the series S (see later) to which it led me. An elementary proof is now obtained by means of Kashikar's Theorem; and thereby we have a new and elementary approach to the properties of the series S.

Corollary 6.

If S (α) denote the series

$$1 + \alpha(\alpha+1)^0 . \frac{x^1}{1!} + \alpha(\alpha+2)^1 \frac{x^2}{2!} + \alpha(\alpha+3)^2 . \frac{x^3}{3!} + \ldots \ldots$$

which is convergent for all values of α if $|x| < \frac{1}{e}$

Using the result in Cor. 5

we get $$S(\alpha) . S(\beta) = S(\alpha+\beta)$$

and therefore $$S(\alpha) = S^{\alpha}$$

where S denotes the series $S(1) \equiv 1 + x + 3\frac{x^2}{2!} + 4^2 . \frac{x^3}{3!} + \ldots \ldots$

Corollary 7.

Since $S^{\alpha} = 1 + \alpha x + \alpha(\alpha+2)^1 \cdot \frac{x^2}{2!} + \alpha(\alpha+3)^2 \cdot \frac{x^3}{3!} + \dots \dots$

we find
$$\underset{\alpha \to 0}{L} \frac{S^{\alpha} - 1}{\alpha} = xS;$$

That is
$$\log S = xS; \quad \dots \quad (1)$$

or
$$S = e^{xS} \quad \dots \quad (2)$$

an important property of S.

[The properties of S, obtained by means of Lagrange's Theorem in the Differential Calculus (from a consideration of the relation $y = xe^y$) formed the basis of my original solution : as they also form the basis of the solution given by Prof. G. N. Watson (who gives a reference to Legendre) and of two solutions given by Mr. Narasinga Rao. Relation (2) above, when we write $y = xS$ transforms into $y = xe^y$, *the basic relation* upon which the investigation by the Differential Calculus proceeds].

V. RAMASWAMI AIYAR.

On Burnside's Condition for the Concurrence of Three Normals to a Conic.

Burnside's condition for the concurrence of the normals at three points whose eccentric angles are α, β, γ, on the conic S, given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

is

$$\sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta) = 0.$$

It can be proved from other considerations that the normals at α, β, γ concur when and only when, $\alpha\beta\gamma$ is a self-polar triangle (or a circumscribed triangle) of some conic coaxal with S — a property, which does not appear to have been noticed. I shall here shew directly that Burnside's condition expresses this fact.

Let
$$Ax^2 + By^2 + C = 0,$$

represent some conic coaxal with S. The conditions that $\alpha\beta\gamma$ may be a self-polar triangle of this conic are :

$$Aa^2 \cos\beta \cos\gamma + Bb^2 \sin\beta \sin\gamma + C = 0$$

$$Aa^2 \cos\gamma \cos\alpha + Bb^2 \sin\gamma \sin\alpha + C = 0$$

$$Aa^2 \cos\alpha \cos\beta + Bb^2 \sin\alpha \sin\beta + C = 0.$$

Eliminating Aa^2, Bb^2, C, we obtain :

$$\Sigma \sin 2\alpha \sin(\beta - \gamma) = 0.$$

Now

$$\begin{aligned}
&\Sigma \sin 2\alpha \sin(\beta - \gamma) \\
&\quad = \tfrac{1}{2}\Sigma\{\cos(2\alpha - \beta + \gamma) - \cos(2\alpha + \beta - \gamma)\} \\
&\quad = -\Sigma\{\sin(\gamma + \alpha)\sin(\alpha - \beta) + \sin(\gamma - \alpha)\sin(\alpha + \beta)\} \\
&\quad = -\{\Sigma \sin(\beta - \gamma)\}\{\Sigma \sin(\beta + \gamma)\},
\end{aligned}$$

since $\Sigma \sin(\beta + \gamma)\sin(\beta - \gamma) = 0$. Thus the condition that $\alpha\beta\gamma$ be self-polar with respect to some coaxal conic reduces to :

$$\{\Sigma \sin(\beta + \gamma)\}\{\Sigma \sin(\beta - \gamma)\} = 0.$$

Here the second factor $\Sigma \sin(\beta - \gamma)$ vanishes *only* when two of the points $\alpha\beta\gamma$ coincide, and may be therefore removed, thus arriving at Burnside's condition. To prove this, write $t_1 = \tan \alpha/2$, $t_2 = \tan \beta/2$, $t_3 = \tan \gamma/2$, so that

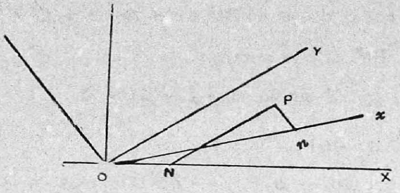
$$\begin{aligned}
\Sigma \sin(\beta - \gamma) &= \Sigma \sin\alpha\,(\cos\beta - \cos\gamma) \\
&= \frac{4}{(1 + t_1^2)(1 + t_2^2)(1 + t_3^2)}\,\Sigma t_1(t_3^2 - t_2^2) \\
&= \frac{-4(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)}{(1 + t_1^2)(1 + t_2^2)(1 + t_3^2)}
\end{aligned}$$

This shews that $\Sigma \sin(\beta - \gamma)$ vanishes, when two of the angles α, β, γ differ by 2π, that is when two of the points $\alpha\beta\gamma$ coincide. Since $t_1 = \infty$ does not make the expression vanish, these are the only cases in which $\Sigma \sin(\beta - \gamma) = 0$.

R. VAIDYANATHASWAMY.

On the Formulæ for Change of Axes.

The following method of obtaining the formulæ for the transformation of axes in plane analytical geometry seems to be new; the nearest approach to it is that given in Jones' *Algebraic Geometry*.



Let OX, OY, and Ox, Oy, be the old and the new axes of co-ordinates and let them contain angles ω, ω' respectively; also let Ox make angle θ with OX. Suppose further that (X, Y) and (x, y) are co-ordinates of any point P referred to the old and new axes.

Draw PN, Pn parallel to OY, Oy meeting OX, Ox in N and n respectively. Now refer the figure to the Argand diagram formed by OX and the straight line through O at right angles to OX. It is then easy to see that

$$\overline{ON} = X$$

$$\overline{NP} = Y\ (\cos\omega + i\sin\omega) = YE\ (i\omega)$$

$$\therefore\ \overline{OP} = \overline{ON} + \overline{NP} = X + YE\ (i\omega). \qquad \ldots \quad (1)$$

Similarly

$$\overline{On} = xE\ (i\theta)$$

$$nP = yE\ (i\theta + i\omega')$$

$$\therefore\ \overline{OP} = xE\ (i\theta) + yE\ (i\theta + i\omega'). \qquad \ldots \quad (2)$$

From (1) and (2), we get

$$X + Y\,E\ (i\omega) = xE\ (i\theta) + yE\ (i\theta + i\omega'). \qquad \ldots \quad (3)$$

By equating the imaginary parts of (3), we have

$$Y\sin\omega = x\sin\theta + y\sin(\theta + \omega'). \qquad \ldots \quad (4)$$

Now multiply (3) by $E(-i\omega)$ and then equate imaginary parts. we thus have

$$X \sin \omega = x \sin(\omega - \theta) + y \sin(\omega - \omega' - \theta) \quad \ldots \quad (5)$$

In the same way, by multiplying (3) by $E(-i\theta)$ and $E(-i\theta - i\omega')$ successively and equating imaginary parts, we get

$$x \sin \omega' = X \sin(\theta + \omega') + Y \sin(\theta + \omega' - \omega) \quad \ldots \quad (6)$$

and

$$y \sin \omega' = -X \sin \theta + Y \sin(\omega - \theta) \quad \ldots \quad (7)$$

When the two systems of axes are rectangular, the equation (3) assumes the particularly simple form

$$X + iY = (x + iy)\, E(i\theta). \quad \ldots \quad (8)$$

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Reviews.

The Great Mathematicians, by H. W. Turnbull. (Methuen & Co.), 128 pages. *Price* 2s. 6*d*.

This is one of a series of excellent popular books on the history of the sciences, which Messrs. Methuen and Co. are publishing. In ' The Great Mathematicians, ' Professor Turnbull has performed his task admirably. Without entering into technicalities which would take away from the popular appeal of the book, he has sought primarily to portray the living personalities of the creators of mathematics, from Thales and Pythagoras down to Ramanujan. The lucid pleasant style in which the book is written, and the innumerable quaint anecdotes which spice its pages will make it a favourite with the public; on the other hand, to the student of mathematics, it has a special and absorbing appeal, as he reads of the far-off situations, in which there first dawned on a human mind some familiar idea or theorem, which he has learned to accept without question, as part of the established order of things.

While there are several ' serious ' treatises on the history of mathematics, Professor Turnbull's ' Great Mathematicians ' is the first book of its kind to be concerned with mathematicians as human beings, and satisfies a crying need, not quite met by a book like Sir Oliver Lodge's ' Pioneers of Science.' It is a book which ought to be in the hands of every student of mathematics, and the author and the publishers are to be congratulated on the great service they have done to the reading public in bringing it out.

R. V.

A TEXT-BOOK OF HIGHER ALGEBRA AND TRIGONOMETRY, by B. B. BAGI, *Dharwar*. Part I., 260 pages; Part II (Infinite Series and Products), pp. 262—368, 1928.

Though the average teacher of Higher Mathematics and particularly of Algebra in Indian Colleges has been well aware of the need for greater rigour in the proofs of theorems and a more precise statement of their limitations, reform in this direction has been exceedingly slow, partly because it requires the study of several treatises, many of them of an advanced character, often in languages other than English, on the part of the teacher, and partly also for want of suitable single Text-books for the use of students. Mr. Bagi's book deserves to be welcomed as an attempt at presenting from the modern viewpoint the revised course in Algebra and Trigonometry leading to the B. A. and the B.Sc. degrees of the Bombay University.

The author omits a treatment of irrational numbers probably because it is considered too difficult for the B. A. student, and relies on geometric intuition for the proof of the theorem that a monotonic bounded sequence tends to a limit Complex Algebra is treated as an Algebra of number pairs, the geometrical aspect receiving considerable attention, and from this the author passes on to De Moivre's theorem and its consequences. A free use of Tannery's theorem for series enables the exponential limit and series and the series for $\sin x$ and $\cos x$ to be obtained with ease.

Features of the book which call for special mention are the detailed references at the ends of chapters to treatises where the topics under consideration are discussed at greater length; the proofs of many theorems by several methods "to cure the students of the superstition that a mathematical theorem can be proved in one and only one way"; the attempt to bring the treatment into relation with the results and methods of the Calculus and Analytical Geometry. The book contains a large number of examples, including many taken from the Bombay University examinations, but quite a large number are original. About 200 examples have been worked out and hints are given for solving many more.

The printing and the general get-up are not bad as judged by the standard of books printed in this country, though the matter is sometimes crowded together and M in an unusual position does duty for Σ. What is more serious is the number of typographical mistakes which a more careful proof-reading could have avoided. An idea of the scope of the book may be gathered from the following table of contents :

Part I. Distribution of products; Determinants; Complex Numbers; De Moivre's theorem and its consequences; Algebraic Equations, *Part I*; Algebraic Equations, *Part II*; Trigonometrical applications of the Theory of Equations; Partial fractions; Summation of finite series; Theory of Infinite series I; Theory of Infinite series II; Special power series; Series of Complex terms; Infinite products.

A. N. R.

Solutions.

Question 258.

(T. SUNDARA RAO, B.A.) :—ABCD is a quadrilateral. A, B, C, D are fixed pivots (fulcra) and C, D are movable points. The quadrilateral is deformed and changes shape. Discuss the curves traced by points on CD and by points outside but rigidly connected with CD.

Partial Solution by S. Srinivasan.

Let a_1, a_2, a_3, a_4 be the lengths of AB, BC, CD and DA respectively, α_2 the angle between AB produced and BC

	α_3	,,	,,	BC	,, ,,	CD
and	α_4	,,	,,	CD	,, ,,	DA.

Projecting on AB and AY perpendicular to AB, we get

$$a_1 + a_2 \cos\alpha_2 + a_3 \cos(\alpha_2+\alpha_3) + a_4 \cos(\alpha_2+\alpha_3+\alpha_4) = 0 \quad \ldots \quad (1)$$

$$a_2 \sin\alpha_2 + a_3 \sin(\alpha_2+\alpha_3) + a_4 \sin(\alpha_2+\alpha_3+\alpha_4) = 0 \quad \ldots \quad (2)$$

Squaring and adding, to eliminate α_4,

$$a_4^2 = a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 \cos\alpha_2 + 2a_1a_3 \cos(\alpha_2+\alpha_3) + 2a_2a_3 \cos\alpha_3 \quad \ldots \quad (3)$$

The co-ordinates of C, D referred to AB and AY as axes are

$$(a_1 + a_2 \cos\alpha_2,\ a_2 \sin\alpha_2)$$

and $(a_1 + a_2 \cos\alpha_2 + a_3 \cos(\alpha_2+\alpha_3),\ a_2 \sin\alpha_2 + a_3 \sin(\alpha_2+\alpha_3))$ respectively.

The co-ordinates (ξ, η) of a point P dividing CD in a given ratio $p : q$ are therefore given by the equations

$$\xi(p+q) = a_1(p+q) + a_2(p+q)\cos\alpha_2 + a_3 p \cos(\alpha_2+\alpha_3) \ldots \quad (4)$$

$$\eta(p+q) = a_2(p+q)\sin\alpha_2 + a_3 p \sin(\alpha_2+\alpha_3) \ \ldots \quad (5)$$

Squaring and adding, we get

$$2a_2a_3p\,(p+q)\cos\alpha_3 = (p+q)^2\,[(\xi - a_1)^2 + \eta^2] - a_2^2\,(p+q)^2 - a_3^2p^2 \quad \ldots \quad (6)$$

From (3) and (6)

$$2a_1a_2\cos\alpha_2 + 2a_1a_3\cos(\alpha_2+\alpha_3) = a_4^2 - (a_1^2 + a_2^2 + a_3^2)$$

$$+ a_2^2\frac{p+q}{p} + a_3^2\frac{p}{p+q} - \frac{p+q}{p}\left[(\xi - a_1)^2 + \eta^2\right] \quad \ldots \quad (7)$$

Taking B as origin and AB produced and BY′ perpendicular to AB as axes, the locus of P is got by eliminating α_2, α_3 from (4), (6) and (7).

If we take $x = \xi - a_1$, $y = \eta$, we obtain from these equations

$$\cos\alpha_2 = l_1\,(x^2+y^2) + m_1x + n_1$$

$$\cos(\alpha_2+\alpha_3) = l_2\,(x^2+y^2) + m_2x + n_2$$

$$\cos\alpha_3 = l_3\,(x^2+y^2) + n_3$$

l's, m's and n's being functions of a's and p and q only.

But $[\cos(\alpha_2+\alpha_3) - \cos\alpha_2\cos\alpha_3]^2 = (1-\cos^2\alpha_2)(1-\cos^2\alpha_3)$

i.e., $\cos^2\alpha_2 + \cos^2\alpha_3 + \cos^2(\alpha_2+\alpha_3) = 1 + 2\cos\alpha_2\cos\alpha_3\cos(\alpha_2+\alpha_3)$

∴ the locus of P is given by

$$[l_1(x^2+y^2)+m_1x+n_1]^2 + [l_2(x^2+y^2)+m_2x+n_2]^2 + [l_3(x^2+y^2)+n_3]^2$$

$$= 1 + 2\,[l_1(x^2+y^2)+m_1x+n_1]\;[l_2(x^2+y^2)+m_2x+n_2]\;[l_3(x^2+y^2)+n_3]$$

which is a sextic of the form

$$r^6 + r^4(p_1+p_2x) + r^2(p_3+p_4x+p_5x^2) + p_6x^2 + p_7x + p_8,$$

r^2 being x^2+y^2 and p_1, p_2, etc., being functions of a's and p and q.

Question 784.

(S. RAMANUJAN):—If $F(x)$ denotes the fractional part of x (*e.g.* $F(\pi) = 0.14159 \ldots$) and if N is a positive integer, show that

(i) $$\lim_{N \to \infty} NF(N\sqrt{2}) = \frac{1}{2\sqrt{2}}, \quad \lim_{N \to \infty} NF(N\sqrt{3}) = \frac{1}{\sqrt{3}},$$

$$\lim_{N \to \infty} NF(N\sqrt{5}) = \frac{1}{2\sqrt{5}}, \quad \lim_{N \to \infty} NF(N\sqrt{6}) = \frac{1}{\sqrt{6}},$$

$$\lim_{N \to \infty} NF(N\sqrt{7}) = \frac{3}{2\sqrt{7}}:$$

(ii) $$\lim_{N \to \infty} N(\log N)^{1-p} F(Ne^{2/n}) = 0,$$

where n is any integer and p is any positive number; show further that p cannot be zero.

Partial Solution by A. A. Krishnaswamy Aiyangar.

(i) Let us consider the general case

$$\lim_{N \to \infty} NF(N\sqrt{a})$$

where a is a positive non-square integer.

If the least positive integer which can be expressed in the form $ax^2 - y^2$ be l, then we know that there are an infinite number of pairs of integral values for x and y, satisfying the equation

$$ax^2 - y^2 = l. \qquad \ldots \quad (1)$$

As the values of x satisfying (1) tend to infinity,

$$\mathop{\mathrm{Lt}}_{x \to \infty} x(x\sqrt{a} - y) = \mathop{\mathrm{Lt}}_{x \to \infty} \frac{lx}{x\sqrt{a} + y}$$

$$= \mathop{\mathrm{Lt}}_{x \to \infty} \frac{l}{2\sqrt{a} - l/x(x\sqrt{a} + y)}$$

$$= \frac{l}{2\sqrt{a}} \qquad \ldots \quad (2)$$

Since $x\sqrt{a} - y = \dfrac{l}{x\sqrt{a} + y}$ and $l < x\sqrt{a}$, we have

$$F(x\sqrt{a}) = (x\sqrt{a} - y).$$

$$\therefore \quad \lim_{N \to \infty} NF(N\sqrt{a}) = \frac{l}{2\sqrt{a}} \text{ from (2) above,}$$

where l is the least positive integer which can be expressed in the form $ax^2 - y^2$.

When $a = 2, 3, 5, 6,$ and 7 l happens to be the difference between a and the greatest square integer less than a; but this is not generally so, for example, when $a = 13$, $l = 1$.

The corresponding values for a and l (up to $a = 19$) are as follows:

a	2	3	5	6	7	10	11	13	14	15	17	19
l	1	2	1	2	3	1	2	1	5	6	1	2

NOTE. (i) l is a quadratic residue of a.

(ii) l is the least of the *divisors* in the odd places, *viz.*, the first, third, fifth, seventh, ... divisors, obtained when converting $\sqrt{a}$ into a simple continued fraction. It is well-known that these divisors form a cycle and so we have only to pick out the least of the numbers in the odd places of the cycle.* If the number of terms in the cycle is odd, it is evident from the properties of the cycle that the least divisor is 1. Thus, the cycle of divisors for $\sqrt{13}$ is (4, 3, 3, 4, 1); the number of terms in the cycle is odd and the least divisor is 1.

The odd cycle also implies that a is a number which can be expressed as *the sum of two squares*.

(ii) For the second part of the question, I am not able to give a proof, but it is possible to push the analysis to some extent and modify

---

* For $a = 19$, the cycle is (3, 5, 2, 5, 3, 1): the least of the divisors in the odd places is 2; thus $l = 2$.

the problem to an apparently simpler form, which if true, will lead to the desired proof.

Let $e^{2/n}$ be converted into a simple continued fraction; and $\frac{l_r}{m_r}$ be the rth convergent of such a fraction.

Now, if
$$\frac{l_{r+1}}{m_{r+1}} > e^{2/n} > \frac{l_r}{m_r}$$

hen
$$e^{2/n} - \frac{l_r}{m_r} < \frac{1}{m_r m_{r+1}} < \frac{1}{m_r}$$

$$\therefore \quad F(m_r . e^{2/n}) = m_r e^{2/n} - l_r$$

$$\therefore \quad m_r (\log m_r)^{1-p} F(m_r e^{2/n}) = m_r^2 (\log m_r)^{1-p} \left(e^{2/n} - \frac{l_r}{m_r}\right)$$
$$< \frac{m_r (\log m_r)^{1-p}}{m_{r+1}}.$$

Evidently, our problem is solved provided we can prove

$$\frac{m_r (\log m_r)^{1-p}}{m_{r+1}} \to 0 \qquad \dots \quad (\alpha)$$

when r tends to ∞, m_r, m_{r+1} being the denominators of two successive convergents, of $e^{2/n}$.

Further, it is sufficient if we can prove the relation (α) for positive values of $p < 1$. For, when $p = 1$, that

$$\frac{m_r}{m_{r+1}} \to 0$$

is easily verified, and when $p > 1$,

$$\frac{m_r}{m_{r+1} (\log m_r)^{p-1}} \to 0, \quad \text{since } \frac{m_r}{m_{r+1}} \to 0$$

and
$$(\log m_r)^{p-1} \to \infty \text{ as } r \to \infty$$

Thus, finally we can prove Ramanujan's problem, provided the following Lemma is true.

Lemma. If m_r, m_{r+1} be the denominators of two successive convergents of $e^{2/n}$ expressed as a simple continued fraction, then

$$\frac{m_r (\log m_r)^\delta}{m_{r+1}} \to 0$$

as $r \to \infty$, δ being any positive number < 1.

N.B.—1. Since in the above discussion, we have not made use of any properties of $e^{2/n}$ other than those of m_r and m_{r+1}, *viz.*

$$\frac{m_r}{m_{r+1}} \to 0, \text{ and } \frac{m_r (\log m_r)^\delta}{m_{r+1}} \to 0 \ (\delta < 1) \qquad \ldots \quad (\beta)$$

as r tends to ∞, we may well replace, in Ramanujan's problem, $e^{2/n}$ by any function of the integer n with the continued fraction properties (β).

2. The critical case $p = 0$, I am not at present in a position to verify.

Question 1122

(V. RAMASWAMI AIYAR):—If the joins of four concyclic points A, B, C, D taken in opposite pairs intersect in P, Q, R, prove that the nine-points circle of PQR passes through the centroid of the points A, B, C, D.

Remarks by C. N. Srinivasiengar.

This result which has been obtained in this Journal is given as an example in W. J. M'Clelland's "Treatise on the Geometry of the Circle," p. 201, where it is attributed to Russell.

Question 1387.

(V. RAMASWAMI AIYAR) :—If (x', y') lie on the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

show that the pedal circle of (x', y') with respect to every triangle inscribed in the conic passes through (ξ, η) given by

$$(a + b)\, x' = (a + b)\, \xi - 2\,(a\xi + h\eta + g)$$
$$(a + b)\, y' = (a + b)\, \eta - 2\,(h\xi + b\eta + f)$$

Interpret the result when $a + b = 0$.

Solution by T. R. Raghavasastri.

If ABC is any triangle inscribed in the conic and P the point (x', y') on it, the pedal circle of P with respect to ABC is the auxiliary circle of the in-conic of ABC with P as focus. If the circular lines though P meet the given conic in Q, R, then PQR and ABC are two triangles inscribed in the given conic with five of the sides of the two triangles touching the in-conic of ABC with P as focus. Hence QR also must touch the in-conic. Therefore the foot of the perpendicular from P on QR lies on the pedal crcle. This is the point required.

Changing the origin to (x', y') the equation of the conic is

$$ax^2 + 2hxy + by^2 + 2x\,(ax' + hy' + g) + 2y\,(hx' + by' + f) = 0.$$

If $x \cos\alpha + y \sin\alpha - p = 0$ is the equation of QR now, PQ and PR are the lines

$$p\,(ax^2 + 2hxy + by^2) + 2\,(x \cos\alpha + y \sin\alpha)\,[x\,(ax' + hy' + g) + y\,(hx' + by' + f)] = 0$$

These are circular lines if

$$p\,(a - b) + 2 \cos\alpha\,(ax' + hy' + g) - 2 \sin\alpha\,(hx' + by' + f) = 0$$

and $$hp + \sin\alpha\,(ax' + hy' + g) + \cos\alpha\,(hx' + by' + f) = 0$$

from which we have

$$2\,(ax' + hy' + g) = -\,2\,hp \sin\alpha - p\,(a - b) \cos\alpha$$
$$2\,(hx' + by' + f) = -\,2\,hp \cos\alpha + p\,(a - b) \sin\alpha.$$

Remembering that $p \cos\alpha$ and $p \sin\alpha$ are equal to $\xi - x'$ and $\eta - y'$ we get

$$2h\,(\eta - y') + (a - b)\,(\xi - x') = -\,2\,(ax' + hy' + g)$$
$$2h\,(\xi - x') - (a - b)\,(\eta - y') = -\,2\,(hx' + by' + f)$$

which are slight modifications of the required equations.

In the case where $a + b = 0$, we have the interesting result that if P is a point on a rectangular hyperbola circumscribing ABC, the pedal circle of P with respect to the triangle ABC passes through the centre of the hyperbola.

Question 1396.

(V. RAMASWAMY AIYAR) :—Given four concyclic points (O_1, O_2, O_3, O_4), if the orthocentres of the triangles whose vertices they are, be denoted by (H_1, H_2, H_3, H_4) in the appropriate manner, show that the set of points (H_1, H_2, H_3, H_4) as well as the six sets of points like (O_1, O_2, H_3, H_4) lie on circles equal to the circle (O_1, O_2, O_3, O_4).

Remarks by C. N. Srinivasiengar.

The earlier part of the question, at any rate, is well-known. *Vide* Askwith's *A Course of Pure Geometry*: Miscellaneous Exercises, example 1. The following properties in connection with the configuration of the eight points are given here for the sake of reference. The reader will easily supply the proofs with the help of the solution on p. 62, *J.I.M.S.*, Vol. XVIII, No. 3.

(1) The relation between the sets of points (O_1, O_2, O_3, O_4,) and (H_1, H_2, H_3, H_4) is reciprocal.

(2) The lines O_1H_1, O_2H_2, O_3H_3, O_4H_4 are concurrent and bisect each other.

(3) The quadrilaterals $O_1O_2O_3O_4$ and $H_1H_2H_3H_4$ are congruent and anti-homothetic figures.

(4) The anti-homothetic centre is the centre of the unique rectangular hyperbola passing through the eight points.

(5) It is also the mid-point of the line joining the centroids of the two quadrilaterals, also of the common chord of the two circles.

(6) It is also the point of concurrence of the nine-points circles of the eight triangles formed by the two sets of points O_1, O_2, O_3, O_4, and H_1, H_2, H_3,H_4.

(7) It is the point of concurrence of the pedal lines of any point of either quartette with respect to the triangle formed by the other three.

(8) The "diameters" of the complete quadrilaterals determined by the sides of $O_1O_2O_3O_4$ and $H_1H_2H_3H_4$ are parallel.

(9) Each of the pairs of quadrilaterals (i) $O_1O_2H_3H_4$, $O_3O_4H_1H_2$. (ii) $O_1O_4H_2H_3$, $O_2O_3H_1H_4$, (iii) $O_1O_3H_2H_4$ $O_2O_4H_1H_3$ is such that the vertices of one quadrilateral are the orthocentres of the triangles formed by the other. The four pairs of quadrilaterals possess the same anti-homothetic centre.

(10) The axes of the parabolas through O_1, O_2, O_3, O_4 are parallel to the axes of the parabolas through $H_1H_2H_3H_4$.

(11) $O_1H_1^2 + O_2H_2^2 + O_3H_3^2 + O_4H_4^2 = 4$ (square on the diameter of any of the circles. (Use (4)).

Question 1465.

(K. V. VEDANTAM):—If the focus of an in-conic of a triangle lies on the polar circle, its directrix passes through the orthocentre.

Solution (1) *by G. V. Krishnaswami and K. Kuppuswami.*

(2) *by Budh Ram Aggarwal, S. V. Minakshisundram, B. Ramamurthi, M. V. Seshadri and N. P. Subramanyan.*

(1) Let the in-conic be S and the polar-circle Σ. Let P be the focus of S lying on Σ. Since the triangle ABC is circumscribed to S and self-conjugate with respect to Σ, there exist triangles inscribed in Σ and self-conjugate to S. Let PQR be one such triangle. Since PQ, PR are conjugate lines with respect to S, they are at right angles so that QR is a diameter of Σ. But QR is the directrix of S. Hence the result.

(2) Reciprocating with respect to P, let $A'B'C'$ be the reciprocal of the triangle ABC. Then the in-conic S goes into the circum-circle S' of $A'B'C'$ and Σ into a parabola Σ' self-conjugate with respect to $A'B'C'$. The directrix of S reciprocates into the centre of S' and the ortho-centre of ABC, *i.e.*, the centre of Σ into the directrix of Σ'. We know, by Gaskin's theorem, that the directrix of Σ' passes through the circum-centre of S'. Hence, in the original figure, the centre of Σ lies on the directrix of S.

Question 1478.

(T. R. RAGHAVASASTRI) :—A circle cuts a parabola in four points A, B, C, D. The tangents to the parabola at A, B, C, D, taken two by two, together with the corresponding chords of contact, give six triangles Show that the circum-centres of the six triangles and the circum-centres of the four triangles formed by any three of the four tangents lie on a circle through the focus of the parabola.

The question as proposed is incorrect. It ought to read thus:

"A circle cuts a parabola in four points. The circum-centres of the six triangles formed by the tangents at the points of intersection, taken two at a time, and the corresponding chords of contact, lie on a circle passing through the focus of the parabola; and the circle through the circum-centres of the four triangles formed by the four tangents is the reflexion of the former circle in the axis of the parabola."

Proof. If the point P on the parabola is $(at^2, 2at)$ the angle ASP is 2θ where $t = \tan\theta$. The equation of the circle through the four circum-centres of the four triangles formed by the tangents at 2α, 2β, 2γ, 2δ on $2a/r = 1 + \cos\theta$ is

$$2r\cos\alpha\cos\beta\cos\gamma\cos\delta = a\cos(\theta - \overline{\alpha + \beta + \gamma + \delta}) \quad (1)$$

[Casey's *Anal. Conics*, page 190].

This can be reduced to the form (focus as origin)

$$x^2 + y^2 - \frac{ax}{2}\left[1 - \frac{2}{a}(2a + g) + \frac{c}{a^2}\right] - 2fy = 0 \quad \ldots \quad (2)$$

if the four points are on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Again the circumcentre of the triangle formed by the tangents at t_1 and t_2 with their chord of contact is

$$\left[\frac{a}{2}(t_1 + t_2)^2,\ \frac{a}{2}(t_1 + t_2)(1 - t_1t_2)\right]$$

with focus as origin.

After some reduction these six circum-centres are shown to lie on the circle

$$x^2 + y^2 - \frac{ax}{2}\left[1 - \frac{2}{a}(2a + g) + \frac{c}{a^2}\right] + 2fy = 0 \quad \ldots \quad (3)$$

Comparing the equations (2) and (3) we find they are reflexions in the axis of the parabola.

Question 1489.

(S. SIVASANKARANARAYANA PILLAI) :—Examine the convergence of the following series:

$$\sum_{3}^{\infty} \frac{\phi(n)}{n^2 \log n \log \log n}$$

where $\phi(n)$ denotes the number of integers less than and prime to n.

Solution by S. D. Chowla.

Let $S(x) = \sum_{1}^{x} a_n$, then we know that

$$\sum_{2}^{x} a_n b_n = \sum_{2}^{x} S(n)\{b_n - b_{n+1}\} + S(x) b_{x+1} - b_2 S_1 \quad \ldots \quad (1)$$

We also have

$$\sum_{1}^{x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x) \quad \ldots \quad (2)$$

Put $a_n = \phi(n)$, $b_n = \frac{1}{n^2}$ in (1). Using (2) we obtain :—

$$\sum_{1}^{x} \frac{\phi(n)}{n^2} = \frac{6}{\pi^2} \log x + O(1) \quad \ldots \quad (3)$$

Now put $a_n = \frac{\phi(n)}{n^2}$ and $b_n = \frac{1}{\log n \log \log n}$ in (1).

Since in this case

$$(b_n - b_{n+1}) \log n \log (n+1) \log \log n \log \log (n+1)$$

$$= \left\{ \log n + \frac{1}{n} + \ldots \right\} \left\{ \log \log n + \frac{1}{n \log n} + \ldots \right\} - \log n \log \log n$$

$\sim \frac{\log \log n}{n}$, we get from (1) and (3),

$$\sum_{3}^{x} \frac{\phi(n)}{n^2 \log n \log \log n} \sim \sum_{1}^{x} \left\{ \frac{6}{\pi^2} \log n . \frac{\log \log n}{n} \frac{1}{\log^2 n (\log \log n)^2} \right\}$$

$$\sim \sum_{2}^{x} \frac{6}{\pi^2} \frac{1}{n \log n \log \log n} \sim \frac{6}{\pi^2} \log \log \log x$$

from the Euler-Maclaurin Sum Formula.

Hence the given series diverges.

Question 1490.

(S. SIVASANKARANARAYANA PILLAI) :—Examine the following statements:—

(i) Every prime factor of $m! + 1$ is congruent to 1 (mod. m).

(ii) If there is no odd perfect number, then no number can be equal to $1\frac{1}{2}$ times the sum of its divisors.

(A perfect number is one which is equal to the sum of its divisors, the number itself excluded).

Remarks by S D. Chowla

There are at least two exceptions to the first statement. For, we have

$$
\begin{aligned}
6! &\equiv 7 \pmod{23} \\
7! &\equiv 3 \;(\;,, \;) \\
8! &\equiv 1 \;(\;,, \;) \\
9! &\equiv 9 \;(\;,, \;) \\
10! &\equiv -2 \;(\;,, \;) \\
11! &\equiv 1 \;(\;,, \;) \\
12! &\equiv 12 \;(\;,, \;) \\
13! &\equiv 18 \;(\;,, \;) \\
14! &\equiv -1 \;(\;,, \;)
\end{aligned}
$$

Thus $14! + 1 \equiv 0$ (mod. 23), but $23 \not\equiv 1$ (mod. 14).

Again, it is easy to verify that

$$18! + 1 \equiv 0 \pmod{23}.$$

and since $23 \not\equiv 1$ (mod. 18), we have another exception.

It would be interesting to know more about this.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1566. (A. A. KRISHNASWAMI AIYANGAR, M.A., L.T.):—Prove by direct algebra that the determinant of the sixth order

$$\begin{vmatrix} x-a, & y-b, & \xi\cos\theta, & \xi\sin\theta, & 0, & 0 \\ x-a', & y-b', & \eta\cos\phi, & \eta\sin\phi, & 0, & 0 \\ x-a'', & y-b'', & \zeta\cos\psi, & \zeta\sin\psi, & 0, & 0 \\ 0, & 0, & x-a, & y-b, & \xi\cos\theta, & \xi\sin\theta \\ 0, & 0, & x-a', & y-b', & \eta\cos\phi, & \eta\sin\phi \\ 0, & 0, & x-a'', & y-b'', & \zeta\cos\psi, & \zeta\sin\psi \end{vmatrix} = 0$$

represents a pair of straight lines.

1567. (A. A. KRISHNASWAMI AIYANGAR, M.A., L.T.):—If $[x]$ denotes the greatest integer in x, and n is any positive integer, show that

(i) $$\left[\frac{n}{6}\right] + \left[\frac{n}{24} + \frac{1}{3}\right] + \left[\frac{n}{24} + \frac{2}{3}\right]$$
$$= \left[\frac{n}{24} + \frac{1}{4}\right] + \left[\frac{n}{24} + \frac{1}{2}\right] + \left[\frac{n}{24} + \frac{3}{4}\right] + \left[\frac{n}{8}\right].$$

(ii) $$\left[\frac{1}{2} + \sqrt[4]{n + \frac{1}{16}}\right] = \left[\frac{1}{2} + \sqrt[4]{n + \frac{1}{2}}\right];$$

(iii) $$\left[\frac{1}{2} + \sqrt[5]{n + \frac{13}{32}}\right] = \left[\frac{1}{2} + \sqrt[5]{n + \frac{7}{16}}\right].$$

1568. (B. RAMAMURTHI):—ABC is a triangle inscribed in a conic A′, B′, C′ are respectively the harmonic conjugates of A with respect to B and C, of B with respect to C and A, and of C with respect to A and B. P is any point on the conic and P_1, P_2, P_3 are respectively the harmonic conjugates of P with respect to (A, A′), (B, B′), and (C, C′) Show that (A, P_1), (B, P_2), (C, P_3) belong to an involution of which the double points are the intersections with the conic of the polar axis of P with respect to ABC. Show also that (A′, P_1), (B′, P_2) and (C′, P_3) belong to an involution and that the Hessian pair of ABC is the common pair of the two involutions.

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LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of April and May 1930.

1 American Mathematical Monthly, **37**, 2, 4.
2 Astrophysical Journal, **70**, 5, **71**, 1.
3 Bulletin of the American Mathematical Society, **36**, 3, 4.
4 Bulletin des Sciences Mathematiques, **54**, Feb. and Mar, 1930,
5 Bulletin de la classe Sciences Physiques et Mathematiques, **4**, 3.
6 Japanese Journal of Mathematics, **6**, 4.
7 Mysore Half-yearly Journal, **3**, 2.
8 Mathematische Annalen, **103**, 1.
9 Messenger of Mathematics, **58**, 9, 10. (2 copies each).
10 Monthly Notices of the Royal Astronomical Society, **90**, 4.
11 Nieuw Archief voor Wiskunde, **16**.
12 Philosophical Magazine, **9**, 57, 58.
13 Proceedings of the London Mathematical Society, **30**, 7. (2 copies).
14 Proceedings of the Physico-Mathematical Society of Japan, **12**, 2, 3.
15 Proceedings of the Royal Society of London, **126**, 803 (2 copies). **127**, 804.
16 Rendiconti del Circolo Mathematico de Polermo, **53**, 2, 3.
17 Revista Mathematica Hispano Americana, **5**, 6, **5**, 1, **3**, 1, 2.

Books.

1 Calendar of the University of Madras for 1929-30.
Contribucion al Estudo de Las Ciencias, No. 89.
...e Magnetic Properties of Atomic Iodine, by Dale Friend Babcock-
...dies in tryptophane metabolism, by Clarence Peter Berg.
Substituted aryl alkyl barbituric acids, by Euclid Wilfred Bousquet.
6 Electrical properties of molecules by the molecular ray method, by Michael Joseph Copley.
7 The synthesis of Cyclobutylmethyl alkyl acetic acids and their bactericidal action toward Beleprae, by Stanley Griffith Ford.
8 Temperature-conductance curves of solid lithium salts at high temperatures, by Defoe Chidress Ginnings.
9 The Electrometer method for measuring dielectric constants of several electrolytes, by Kenneth Orville Smith.
10 The synthesis of certain octadecanoic and related acids and their bactericidal action toward mycobacterium leprae, by Wendell Meredith Stanley.
11 Physico-Chemical Studies on the mechanism of the drying of linseed oil, by Herman Louis Tschentke.

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